The Artistic (and Practical) Utility of Hyperspace

I have felt and given evidence of the practical utility of handling space of four dimensions as if it were conceivable space. -James Joseph Sylvester, 30 December 1869

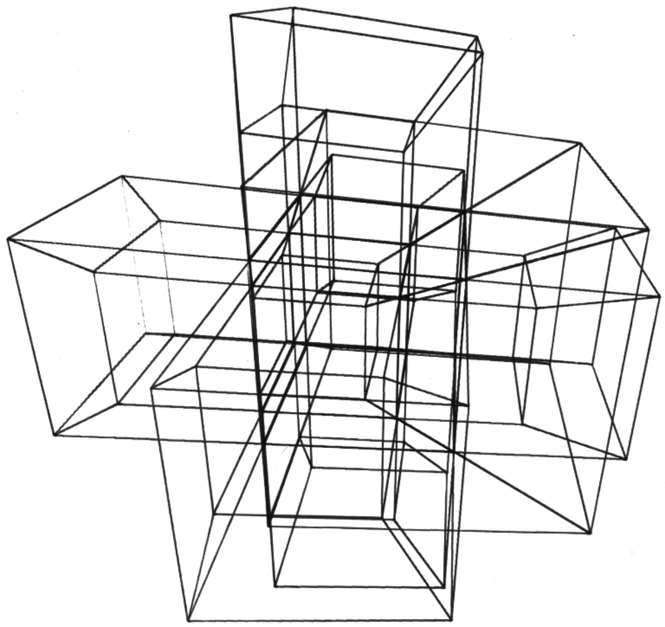
The best way to show how four-dimensional geometry has enriched my artwork is to show how my artwork has enriched four-dimensional geometry. Of course, I had teachers.

Hypercubes Tessellated

In the summer of 1979, when I was 35, I traveled to Brown University to meet Tom Banchoff, chair of the mathematics department, and to see his computer representation of a hypercube rotating in four-dimensional space. Banchoff was generous with his time, and with time on his million-dollar VAX computer. Subsequent visits, hand-written letters, which I have cherished and kept, proofs of my conjectures, invitations to conferences, and authentication of my computer programs and of the mathematical content of my work - all this followed.

Do you know the plane if you only know a square? Wouldn't it be better to contemplate a whole page of squares fitted together, a *tessellation* of squares? Likewise, do you know space if you only know a cube? Soon after I visited Banchoff for the first time and learned to replicate his program for the rotating hypercube (at Pratt Institute with Herb Tesser and his million-dollar VAX), I programed *9 Tessellated Hypercubes*, linked here - tonyrobbin.net/quasi/TessHyperCubes.mp4, and see drawing at the end of this section. One hypercube above, one below, to the left, to the right, in front of and in back of, and also one fore and one aft in the fourth dimension were place around a central hypercube. To my knowledge, this was the first time a program was written (and a resulting video was made) that showed tessellated hypercubes rotating in real time about four mutually perpendicular axes taken two at a time. My program also showed the rotations in both isometric and perspective projection, and with the option of viewing in anaglyphic (red and blue) stereo. Further, the hypercubic rotations also included body-centered rotations such as the pitch, roll, and yaw that a pilot would have to control. Seeing the tessellated hypercubes gives a more vivid understand of 4space than seeing just one hypercube.

Squares are tessellated if their boundary edges, are edges in precise two squares; cubes are tessellated if their boundary squares are squares in precisely two cubes; and hypercubes are tessellated if their boundary cells are cells in two and only two hypercubes. Said this way, the tessellation rule is clear, but I found it hard to visualize what Donald Coxeter called a "honeycomb" of hypercubes. With an introduction from Linda Henderson, I wrote to Coxeter, and received back one page of diagram and text that made the problem clear, and I quickly wrote the code. (I have cherished that handwritten correspondence too.) I published pictures of the 9 tessellated hypercubes in my 1992 book *fourfield: Computers, Art & the 4th Dimension* and even included 3D glasses and a print of the tessellation in 3D.



Tony Robbin, *9 Tessellated Hypercubes*, *c.* 1983. Plotter drawing on paper. Collection the artist.

Planar rotations

When I visited Banchoff at Brown that first time, I traded a drawing for a print of his pioneering film T*he Hypercube: Projections and Slicing*, 1978 (https://www.youtube.com/watch?v=90olwwLdEYg) Leaving the slicings aside for the moment, consider the four-dimensional rotations and resulting changes in the projections of the hypercube. A paper square can rotate around a pin stuck into it. A cardboard box of safety matches can rotate around a bamboo skewer shoved through it (an axel). But a hypercube can rotate around a plane. This is a special kind of rotation that can only happen in four or more dimensions; it is a rotation that requires the extra degrees of freedom that 4space can offer. This higher-dimensional rotation is called planar rotation, and without planar rotation one does not really have 4D art.

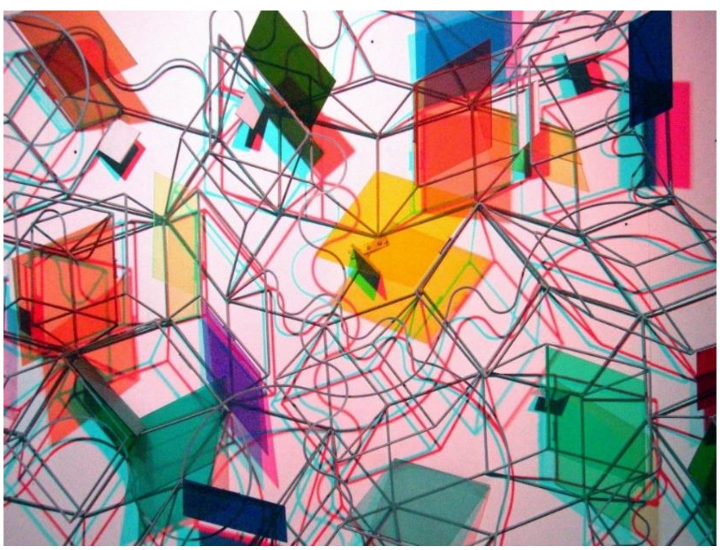
These planar rotations are best understood by looking at the matrix algebra, see appendix 1.

George Gamow's *One, Two, Infinity* was one of the trove of books given to me by John McIlroy when he retired from the math department of Trenton State College where I briefly taught in the 1970s; McIlroy lit a fuse when he challenged me: "Read these books and they will change the way you see!" In that book, Gamow made the odd claim that the Lorentz transformations of special relativity were planar rotations in the fourth dimension. And eventually I did see that the 4x4 matrix of Minkowski's spacetime metric has the form of a planar rotation: push on one thing and another thing changes: make lengths shrink in one of three spatial dimensions and time has to run slower, i. e. the interval between "clicks" has to expand. Two things change but the rest do not. That is, we see a spaceship traveling close to speed of light snubbed in length in the direction it is going, and we also notice its clocks running slow. But the height and width of the spaceship remain the same as it always was. The rubbery, reciprocal nature of spacetime has this formal association with planar rotations in 4space

All the mathematics of 4D rotation, including those in Banchoff's film and my programs too, are planar rotations about the origin – about the center of the hypercube, or the center of the central hypercube in a tessellation. But for my artworks, I wanted to rotate about a plane that is a face of one of the cells of the hypercube. I thought I knew what that would look like, but I went to Brown to ask Banchoff to confirm my understanding on his computer, which he was able to quickly do. It looks different than rotations about the origin.

My works from the 1980s, especially *Fourfield, Lobofour,* and the light pieces use two-dimensional elements (lines painted on a canvas or fixed shadows from colored lights) and three-dimensional elements (thin steel rods welded to become three-dimensional line drawings) that work together to give the visual information of planar rotation. The fixed two-dimensional elements do not change as you walk around, but the three-dimensional elements do parallax. (Some things change, and some do not.) In the light pieces, red and blue colored light reflect white on the wall, but where the red light is blocked a blue line of light appears, and where blue is blocked a red line appears. These colored lines can be combined by red and blue 3D glasses to make an illusion of three-dimensional elements rotating through the steel rod structures for a convincing display of four-dimensional planar rotation.

The wall becomes a face of the hypercube, and the floor becomes the zw plane. Walking on the floor while wearing the 3D glasses you are causing a four-dimensional, planar rotation – you are walking in the fourth dimension.



Tony Robbin, *1987-3*, 1987. Welded steel, acrylic plates, and colored lights. 84 x84 x 8 inches. Collection the artist.

Slices vs. Projections

Banchoff's 1978 film distinguishes between slices and projections of the hypercube. My 2006 book *Shadows of Reality, the Fourth Dimension in Relativity, Cubism, and Modern Thought*, reviews the art and math history of the first few decades of the 20th century (making some new discoveries) and reaches this conclusion: the projection model of the fourth dimension does all the work and the slicing model gets all the credit.

Shadows discusses the flatland model where three-dimensional objects pass through a two-dimensional world in analogy to four-dimensional objects that pass through our three-dimensional world. This model, popularized by Edwin Abbott Abbott's 1884 book *Flatland*, dominated popular thought until well into the 20th century. As Henderson has amply demonstrated in many papers and books, the whole of European culture was fascinated by a hidden reality: atomic structure, what was revealed by the new x-rays, radio waves penetrating everywhere, and things, from atoms to ghosts, that are hidden in the fourth dimension. The flatland or slicing model explained this world beyond.

But soon after the turn of the 20th century, papers by Washington Irving Stringham (1847-1909) and Victor Schlegel (1843-1905), and especially books by Esprit Jouffret (1837-1904) – among others - were passing beyond mathematical circles to enter the general culture as well as artworld. These math papers and books examined the projection model, based on projective geometry, which is intuitively understood as shadows from a higher dimension. My book *Shadows* shows that it is this projection model that Pablo Picasso (1881-1973) used to further his creation of Cubism (perhaps it should be called Hypercubism). Picasso used four-dimensional geometry to free himself from the tyranny of the surface, the skin, to show the psychological reality within. *Shadows* then shows that it is the projection model that is the basis of Hermann Minkowski's (1864-1909) spacetime formalism of Special Relativity. *Shadows* continues by discussing such challenging contemporary physics topics as Twistors, Entanglement, Category Theory, and Quasicrystals. Each of these topics owes far more to the projection model of 4space than to a flatland model; indeed, the flatland model of higher dimensions plunges one into hopeless confusion when thinking about these topics. *Shadows* also rediscovers the remarkable polymath T. P. Hall.[i] . I was motivated to explore the implications of the distinction between slices and projection by Banchoff's pioneering 1978 film.

The relentless assault on commonsense reality provided by new technologies in the early 20th century needed the artist to make the new reality, and the fourth dimension, comprehensible and stable. Heavier-than-air ships could not possibly fly, just think about it! said the famous mathematician Simon Newcomb (1835-1909). But what a mistake it was to ever adopt the more reasonable space+time slicing model of four-dimensional reality over projective spacetime, the less reasonable, more accurate, more giving model of another fungible <u>geometric</u> dimension fully inserted into our familiar width, length, and height. Projective geometry is essential to a vision of an invisible, fecund, immaterial primal extra-dimensional soup that makes reality.

Fat Topology

In 2014, I deconstructed a painting from 2006 for the online journal *Symmetry* in an article called "Topology and the Visualization of Space" https://www.mdpi.com/2073-8994/7/1/32. The journal's special issue "Diagrams, Topology, Categories, and Logic" was guest-edited by topologist Louis Kauffman. I met Kauffman years before when I took a booth at an AAAS meeting to show my quasicrystal sculpture. We had dinner and had a conversation that changed my life. He subsequently invited me to a small conference at the University of Minnesota's Superconducting Computer Center where I met Scott Carter, Jeff Weeks, Charles Gunn and other topologists. This is some of what I learned from Carter: to braid one-dimensional threads you need access to three-dimensional space to pass the threads over and under each other, and to braid sheets, not ribbons but infinite sheets, you also need to have access to two more dimensions than the two-dimensional sheets themselves.

As a semi-pro four-dimensional geometer, unstructured, curving, and braided infinite sheets made me nervous. After renewing my friendship with Carter through invitations to speak at his University beginning in 1998, I took the flat patterns of my previous paintings and swooped them, curved them, interlaced them, and braided them. Like before, each patterned sheet was color-coded with a single color, and like before each sheet was defined by a different geometric pattern. But unlike before, each pattern had a thickness and was defined by three-dimensional polyhedra rather two-dimensional shapes. Fat sheets, fatter and fatter: not planes but hyperplanes, maybe even one could think of them as braided spaces.[ii]

Almost 50 years ago, I read the following passage from Albert Einstein's fifth appendix (1952) to his popular 1916 book *Relativity: The Special and the General Theory*.

> When a smaller box *s* is situated, relatively at rest, inside the hollow space of the larger box *S*, then the hollow space of *s* is a part of the hollow space of *S*, and the same "space", which contains both of them, belongs to each of the boxes. When *s* is in motion with respect to *S*, however, the concept is less simple. One is then inclined to think that *s* encloses always the same space, but a variable part of the space *S*. It then becomes necessary to apportion to each box its particular space, **not thought of as bounded**, and to assume that these two spaces are in Before one has become aware of this complication, space appears as an unbounded medium or container in which material objects swim around. But it must now be remembered that there is an infinite number of spaces, which are in motion with respect to each other. [Dover edition of *The Principle of Relativity*, pp. 138-9, emphasis added].

In my paintings from 1998 onward, braided lattices represent the multiple unbound spaces that Einstein wrote about, and that had a special resonance for me given the way I grew to see the world. Many spaces in the same space.

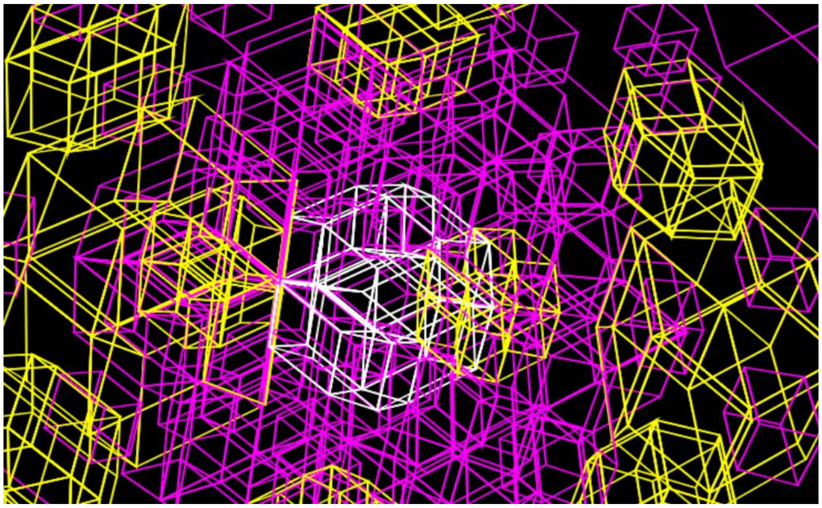
Tony Robbin, *2006-6*, 2006. Acrylic on canvas. 56 x 70 inches. Collection the atist.

Quasicrystals

Reality is in higher dimensions; our experience is but a projection of that higher dimensional reality. Such a statement smacks of religion or at least Platonism. But if the statement can be divorced from those associations, a new understanding of modern physics and the richness of projective geometry can be had.

For those interested in four-dimensional geometry, especially my friends Koji Miyazaki and Haresh Lalvani, quasicrystals had an odd familiarity. I went to Philadelphia to visit Paul Steinhardt to learn more. Steinhardt received me cordially, gave me the thorough papers he had written with his collaborators, and answered my letters as he coached me in learning to code the deBruijn algorithm for quasicrystals. (Nicolaas Govert deBruijn,1918-2012). Later I met deBruijn himself, was a guest in his home, and we stayed up late talking quasicrystals. Steinhardt sought to discover foolproof local matching rules that obviated Roger Penrose's conjecture that, since no local matching rules existed, real, physical quasicrystals must self-assemble according to some quantum-like method.

Soon after I managed to write my Pascal programs for quasicrystals implementing the deBruijn algorithm, I visited George Francis at the University of Illinois/Champaign-Urbana. Francis threw me, without warning, into his honors class in mathematics, and I started talking about quasicrystals. There followed decades of collaboration, in which Francis and his students translated my archaic Pascal to Python and expanded my code to include features I wanted but did not have. Francis put the enhanced program in the "Cube," a virtual environment where one could wander at will through a quasicrystal, peering at its components, and gaining an intuitive feel for the structures otherwise unobtainable.



The Robbin/Francis program can be linked here: http://new.math.uiuc.edu/Tavares/htr4.html

Originally, I wanted to make quasicrystal architecture and sculpture, and did make large pieces for the Danish Technical University and the Jacksonville Library system, as well as for a number of temporary exhibitions. But then, I started thinking more and more about Penrose's conjecture of an application to quantum physics.

Altogether, I spent 35 years studying and programming the deBruijn algorithm for quasicrystals, and recently I have concluded that the algorithm is a geometric model of quantum non-locality. I had been discussing what is known as quantum information theory with Padmanabhan K. Aravind because Aravind was investigating the use of figures from projective geometry to model the fundamental quantum paradox of entangled particles. The deBruijn algorithm is also based on projection: a regular six-dimensional cubic lattice is projected to 3space in just the right way to induce a foolproof potentially infinite quasicrystal. In fact, the algorithm bounces back and forth between these two states: the six-dimensional before, and the three-dimensional after. If the algorithm represents a physical system then it passes back and forth through what should be a one-way, entropy gate. But this is precisely the quantum paradox of entangled particles: impossible, spacelike separations of information are overcome!

I wrote this all up:
https://www.researchgate.net/publication/343099608_The_deBruijn_Algorithm_for_Quasicrysals_as_a_Model_for_Quantum_non-Locality but the problem with making my argument convincing is that one has to get into the difficult nitty-gritty of the algorithm, to get your hands dirty as the mathematicians say. It is challenging to understand just how and why the algorithm works.

Something from Nothing,

The conversation with Kauffman that changed my life was on the topic of how do you get from pure abstraction to physical reality: what John Wheeler called "pre-geometry," and Roger Penrose investigated with spin networks, and Carter, Kauffman, and Masahico Saito called Topological Lattice Field Theory. In analogy to the origin of life, where organic compounds spontaneously arise from inorganic ones fitted into a clay template, the origin of physical existence spontaneously arises from something that is nothing-at-all, and from that nothing-at-all to geometry, and from geometry to spacetime, and from spacetime to everything.

That physics could ask these metaphysical questions, that famous mathematicians begged for these questions to be answered and were working on it, and that I could have a considered opinion – all of this ennobled me. Going toe to toe with Kauffman at dinner at the AAAS meeting in Boston (must have been the one in 1988), gave me the impetus to study and speculate further.

For years and years, I had the notion, intuitive and vague, that four-dimensional space must be curved. Now, almost every mathematician will tell you that this is nonsense: four-dimensional and non-Euclidian are two separate geometries that have nothing to do with one another. And a mathematician might also suggest that an artist's fantasies could not possibly be connected to real thought. But as I later read in Felix Klein's (1849-1925) *Development of Mathematics in the 19th Century*, 1926 (who had a similar problem with his fellow mathematicians), if you happen to be <u>in</u> space, then infinity to the left and infinity to the right are the same point at infinity because in projective geometry there is only one point at infinity. And as Coxeter concludes: "Thus, if metrical ideas are left out of consideration, elliptical geometry is the same as real projective geometry." (1942, p.15)

There must be some set of logical relationships in the universe or else the universe would not be as consistent as it is. John Wheeler calls that set of logical relationships a "pre-geometry." Penrose's "spin-networks" defines that set as tri-valent diagrams of spins of would-be particles, diagrams that assemble to look like space. Penrose's spin networks are like an immaterial version of a mycelium underground that from place to place erupts in a fruiting mushroom. As John Baez writes in an important paper for the arXiv, that concept was powerful to those working on quantum loop theory because of loop theory's "insistence of a back-ground free approach." In other words, spin networks do not happen in space, they make space.

Carter, Kauffman, and Saito were influenced by spin networks when they collaborated on "Topological Lattice Field Theory." I spent a fair amount of time working to understand the theory for my book *Shadows*. Starting from a simple diagram of associations and noting that the associations change if the diagram is rotated or looked at from a different location, the authors build a multidimensional topological structure on which a universe could be built. "Pancher moves [extended] to 4 manifolds" is an argument made with diagrams that climbs the dimension ladder from two dimensions to four, and from logical associations to quantum gravity.

My thoughts about something-from-nothing center on the role of projective geometry in describing, no in <u>making</u>, reality. I learned this from Penrose's writings, and a couple of conversations with him: a light ray is more like a projective point than a line in space. A projective point is what an artist would call the completely foreshortened edge. If I am reading Penrose correctly, the Minkowski metric for spacetime (and special relativity) is the inevitable consequence of space being projective. And as I have argued, it is something like the N/2 projection of the deBruijn algorithm that permits the paradoxical logic of the quantum world. Say that it is the nature of physical space to be projective and all else fall out.

Shape Shifting in the 1970s

An artist's life experiences influence that artist's aesthetics; sometimes art history forgets this. (Could it possibly be the same for a mathematician, even a little?) I grew up in Japan, Kansas City Missouri, Okinawa, Teheran, and spent time in Frankfurt, and Andover before settling in New York. In each place, I was expected to be a different person: a juvenile delinquent, an athlete, a scholar, a hippie, an artist. When I was a professor at Trenton State College, I marveled that many of my students never went anywhere: maybe to the mall in New Brunswick, never to Manhattan. But the world came to them via the movies and television, and they had somewhat of the fluid identity that I needed in order to be welcome in the various societies I grew up in. Just after graduate school at Yale, I met Robert Jay Lifton, who interviewed young Japanese men after the war. He was astonished: where they once worshiped the Emperor, they now embraced Christianity; where they once hated their enemy America, they now loved Americans; where they once cherished their agricultural heritage, they now favored capitalism. Lifton coined the term "proteanism" to describe those young Japanese men, and suggested that their behaviors were admirably adaptive, and further, that identity could be far more fluid than we had been led to believe.

Linda Ronstadt sang country-rock in a boy scout shirt and hot pants, sang Gilbert & Sullivan in a bonnet, 50s torch songs in an evening gown, country-western in jeans, and Mariachi with flowers piled on top of her head. Lee Breur presented Sophocles as an African-American gospel service, and Dante's *Divine Comedy* with Bunraku puppets and Motown music. The Beatles started out as a Mod band from Liverpool playing American Rhythm & Blues, then they transitioned to music influenced by Indian Raga, to the goof of St. Pepper's Lonely Hearts Club Band, and finally to avant garde electronic music.

Thomas Berger's novel of 1964 *Little Big Man*, later a popular film directed by Arthur Penn and starring Dustin Hoffman, 1970, tells the story of a protean man who changes back and forth from White to Native American, usually to avoid threats to his life. At times, he is also a farmer, a gunslinger, a merchant, and a drunk. Hoffman truly inhabits the different identities with differences of costume, speech, and body language. The Dave Brubeck Quartet's biggest hit, and the bestselling jazz single ever, is *Take Five*. The title is not a command to take a short break, but rather is a reference to the non-Western 5/4 time that the quartet heard during a State Department sponsored tour of Turkey in 1958. Paul Desmond, writing for the group, made the most of the quirky rhythm, quirky to Western ears.

I use projective geometry to depict this protean worldview: many spaces in the same space.

And in these worried times, cross-cultural fertilization shapeshifting with its try-on identities is sometimes denigrated as imperialist cultural appropriation. But there is another way to look at it, a way of looking that we need now more than ever: see the commonality of sapiens as exemplified by the universality of pattern. We need to see common biology, not cultural differences. Mathematics is the same for everyone, too.

Finally, in a paraphrase of an adage of mathematical Chaos Theory: a bat shits in WuHan and 300,000 Americans die. The whole world is focused on every point on earth; every point of earth is projected on the whole world.

Tony Robbin, Gilboa 2020, during Covid lockdown

Appendix 1

To rotate a point in the xy plane around the origin by an arbitrary angle (a), and “multiplying on the left”

$$\begin{matrix} \cos(a) & -\ sin(a) \\ \sin(a) & +\ \cos(a) \end{matrix} \;\text{x}\; \begin{matrix} x \\ y \end{matrix} = \begin{matrix} x' \\ y' \end{matrix}$$

If plane xy is in 3-space, then

$$\begin{matrix} \cos(a - \sin(a) & 0 \\ \sin(a) + \cos(a) & 0 \\ 0 \quad\quad 0 & 1 \end{matrix} \;\text{x}\; \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} x' \\ y' \\ z \end{matrix}$$

And this allows for two more rotations:

$$\begin{matrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{matrix} \;\text{x}\; \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} x' \\ y \\ z' \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 0 & 0 \\ 0 & \cos(a) & -\sin(a) \\ 0 & \sin(a) & \cos(a) \end{matrix} \;\text{x}\; \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} x \\ y' \\ z' \end{matrix}$$

Rotation in x,z

rotation in y,z

If plane xy is located in 4-space, then

$$\begin{matrix} \cos(a) & -\sin(a) & 0 & 0 \\ \sin(a)\ + & \cos(a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \;\text{x}\; \begin{matrix} x \\ y \\ z \\ w \end{matrix} = \begin{matrix} x' \\ y' \\ z \\ w \end{matrix}$$

And this allows for a total of 6 rotations: xy, xz, xw, yz, yw, zw

[i] My hero, the great polymath Thomas Proctor Hall (1858-1931) studied projective models of four-dimensional geometry to master his image of reality: as a physician, that x-ray could not only diagnose, they could cure; as a mathematician, that dynamic, telescoping and hinged glass-tube models of hypercubes could show how planar rotation works 75 years before it was seen on the computer screen; and a science fiction writer, that stories of trans-material essential beings could teach us about the divine. He was, by turns, chemist, physicist, mathematician, physician, and writer. His life should be better known, but I think there are some secrets that cancelled his fame. I still want to write his biography.

[ii] How lucky to have discovered the programming language *Formian* just at this time. Developed by Hoshyar Nooshin at the University of Sussex at Guildford for the civil engineering department, *Formian* is a language of pattern generation, in an arbitrary number of dimensions, with many procedures to curve surfaces and lattices. I was invited to Guilford for a two-week seminar to learn this software and was presented with several versions of the language.

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Acknowledgements

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I must also praise two private tutors. John Swartz who patiently walked me through *Gravitation* by Misner, Thorne, and Wheeler. And Charles Scheim who helped me understand projective geometry when I was working on *Shadows.*