Preface to *Shadows of Reality, The Fourth Dimension in Relativity, Cubism, ad Modern Thought*

A Yale University Book, 2006

We walk in the here and now, but is there a space beyond, a space that impinges on our own infinite space, or more dramatically, a space wholly applied to or inserted into our space? Perhaps we remember being in the space of the womb and then we remember the cold infinity of space that exists after our birth, and those memories foster our belief that such a space beyond space is possible. Can mathematics define and conquer the extra space of space and make four-dimensional geometry into a sensible world, even as sensible as the three-dimensional world? During the nineteenth century, mathematicians and philosophers explored and comprehend such difficult thoughts by the use of two mathematical models: the Flatland or slicing model and the shadow or projection model. My thesis is that the projection model does all the work (of explaining things), but the slicing model gets all the credit.

We can understand these two metaphors of four-dimensional space by considering different two-dimensional manifestations of a chair. The Flatland model assumes viewers to be pond scum, floating on the surface of the water. As the chair slips into their world, successive slices of the chair are wetted. First the four legs appear as four circles; then, the seat is a square; then, two circles again as the back approaches the water; and finally, the thin rectangle of the back of the chair is present in the two-dimensional world. But in the shadow model, if the sun were to cast a shadow of the chair on the surface of a smooth beach, then the whole chair would be present to the two-dimensional creatures living on that beach. True, it could happen with shadows that lengths or angles between the parts could be distorted by the projection, but the continuity of the chair is preserved, and with that is preserved the relationship between its parts.

The strength of the slicing model is its grounding in calculus, which reinforces the notion that slices represent reality by capturing infinitely thin sections of space and then stacking them together to define motion. Further, the stacking together of all of space at each instant is a definition of time; almost everybody knows that time is the fourth dimension. The slicing model is mathematically self-consistent and thus true, and it is often taken to be an accurate, complete and exclusive representation of four-dimensional reality. This may seem to be the end of the story, yet the Flatland metaphor constrains thought as much as it liberates it.

The projection model is an equally clear and powerful structural intuition that was developed at the same time as the slicing model. Contrary to popular exposition, it is this other model, the projection model, that revolutionized thought at the beginning of the twentieth century, and the ideas developed as part of this projection metaphor continue to be the basis for the most advanced contemporary thought in mathematics and physics. Like the slicing model based on calculus, the shadow model is also self-consistent and mathematically true; it is supported by projective geometry, an elegant and powerful mathematics that, like calculus, also reached a flowering in the nineteenth century. In projective geometry a point at infinity lies on a projective line, is a part of that line, and this simple adjustment of making infinity a part of space vastly changes and enriches geometry to make it more like the way space really is. Projected figures are whole, sliced figures are not, and more and more the disconnected quality of the Flatland spatial model presents problems. Even time cannot be so simply described.

Picasso not only looked at the projections of four-dimensional cubes in a textbook when he invented Cubism, he also read the text, as he embraced the ideas and not just the images. Hermann Minkowski had the projection model in the back of his mind when he used four-dimensional geometry to codify special relativity; a close reading of the texts shows this to be true. Nicolas De Bruijn's projection algorithms for generating Quasicrystals revolutionized the way mathematicians think about patterns and lattices, including the lattices of atoms that make matter solid. Roger Penrose shows that a light ray is more like a projected line than a regular line in space, and the resulting Twistor program is the most provocative and profound restructuring of physics since Einstein. Projective geometry is now being applied to the paradoxes of Quantum Information Theory, and projections of regular four-dimensional geometric figures are being observed in quantum physics in a most surprising way. We use projection methods to climb the dimension ladder in order to study Quantum Foam, the exciting and most current attempt to understand the space of the quantum world. Such new projection models present us with an understanding that cannot be reduced to a Flatland model without inducing hopeless paradox. These new applications of the projection model happen at a time when computer graphics gives us powerful new moving images of higher-dimensional objects. The computer revolution in visualization of higher-dimensional figures is presented in Chapter 10.

Projective geometry began as artists' attempt to create the illusion of space and three-dimensional form on a two-dimensional surface. Mathematicians generalized these perspective techniques to study objects in any orientation and eventually in any number of dimensions, thus establishing a generalized projection. Simultaneously, projection evolved to projectivity, whereby objects and spaces were studied with an eye to what remained constant, as structures were passed from pillar to post by chains of projection operations, including those projected object back onto themselves. Finally projective came to mean systems defined by homogeneous coordinates where the concept of metric dimension loses all traditional meaning, but gains a richness relevant to modern understanding. Perspective, projection, projectivity, projective - these subtle concepts promoted one another, building higher levels of abstraction, until self-referential, internally cohesive structures are defined that are housed in a higher-dimensional framework. Such higher-dimensional frameworks now begin to have more and more reality as they become more familiar, and as culture stabilizes their appearance.

I have been on this journey for over thirty years. For this book, I looked back with pleasure to the time when the projection model of four-dimensional geometry first appeared. I got to know Washington Irving Stringham better, the nineteenth century mathematician whose drawings of four-dimensional figures caused a sensation in Europe and America. I discovered the amazing Dr. Hall who anticipated by 75 years the behavior of computer-generated four-dimensional figures. I saw the moment when Picasso invented true Cubism, and without this backward look I never would have met the wonderful Alice Princet, Picasso's partner in his four-dimensional quest. I always wanted to know Minkowski's mindset better. It was fun rutting around in the dusty stacks of the Columbia University Mathematics Library and the Clark University Archives, and I am grateful for new e-mail pals, archivists in the United States and Europe.

Even more thrilling was talking with living mathematicians and physicists, deepening old friendships and making new ones. Many of the people in the later chapters of this book made time for me out of a respect for my artwork, my pioneering computer-programing of the fourth dimension, and my commitment to visualizing four-dimensional geometry. Their acceptance of me, and the access they consequently provided, make this long writing project worthwhile. I got and also gave.

Tony Robbin, Gilboa and New York